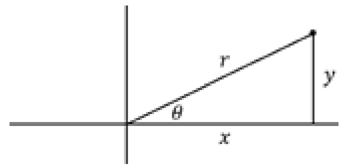
Polars

Polar points are representations of the same points as rectangular coordinates the are just different routes to arrive at the same points.

Imagine two super heroes. One that gets around primarily by vehicle or by utilizing buildings and another that can fly.

The hero who is restricted by city blocks might describe a route as 3 miles east and 4 miles north. The hero that can fly would describe the same trip as 5 miles at a specific bearing (0.972 radians north of east).

This is the essentials of polar vs rectangular coordinates.



$(x, y) = (r, \theta)$

$x = r\cos\theta$ $\quad$ $r = \sqrt{x^2 + y^2}$ (A negative value of r indicates travel opposite of the directional ray.)

$y = r\sin\theta$ $\quad$ $\theta = \tan^{-1}\left(\frac{y}{x}\right) \pm k\pi$ (k is an integer and is a result of arctangent's restricted range.)

Much like parametric equations and vectors, we may take advantage of the chain rule to find the slope of the tangent line along our curve. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$. So, using product rule:

$y = r\sin\theta$ $\quad$ $x = r\cos\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$ $\quad$ $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

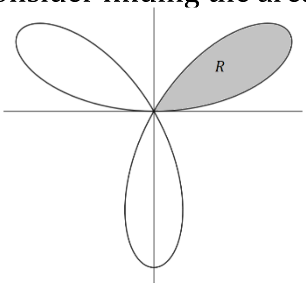
And thus, $\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$.

Students do not need to memorize this form. They may develop it as needed using the product rule and substituting known values before evaluating the quotient $\frac{dy/d\theta}{dx/d\theta}$.

Arclengths of polar curves are not found for the AP® Calculus exam, but it may be taught, time permitting, since the formula is relative easy and gives some practice working with trigonometric integrals and/or calculator integration. (Arclength of a polar curve $= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.)

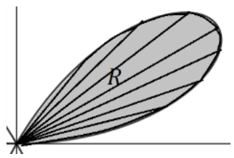
Polar area is a great integral application that helps students (and teachers) grasp the radial nature of polar graphs and yet again look at how Riemann sums are limits of infinite sums.

Consider finding the area enclosed by one petal of the polar curve $r = \sin(3\theta)$?



The area cannot be sliced into rectangles since the defining equation is not rectangular.

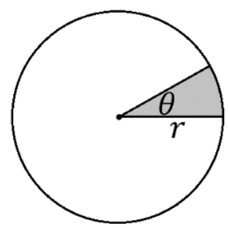
To approximate the area, note that the function defines rays inside the curve and we want to find the area between individual rays.



Instead of approximating by rectangles under the curve, we can approximate with sectors of circles within the curve.

Recall sector area:

Portion of circle: $\frac{\theta}{360^\circ} = \frac{\theta}{2\pi}$



Area of circle: $A = \pi r^2$

Area of portion: $A_{sec} = \frac{\theta}{2\pi}\,\pi r^2 = \frac{1}{2}\theta r^2$

So the polar area can be estimated with:

$$\sum_{k=1}^{n} \frac{1}{2}[r(k)]^2 \Delta\theta$$

And as done in the past, evaluate more and more sectors. $\Delta\theta = d\theta$ will become infinitely small as we add up infinitely thin sectors of our area.

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}[r(k)]^2 \Delta\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Many students wonder why the equation is without a π since the origins are circular sectors. The area may be presented as the following to emphasize both the connection to circle area and the sectors/portions:

$$\text{Area} = \frac{1}{2\pi}\int_{\alpha}^{\beta} \pi r^2 d\theta$$

Sometime must be spent with polars to get comfortable with the radial nature of the curves. Returning to our original problem, $\sin(3\theta) = 0$ the first two solutions are 0 and $\pi/3$. The area of this petal is given by $\frac{1}{2}\int_0^{\pi/3} r^2 d\theta$.

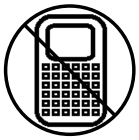
While students will generally do these problems with graphs provided or calculators allowed, they should be familiar with common polar graphs and features of sine and cosine. In particular, sine graphs are symmetric with the vertical axis and cosine graphs are symmetric to the horizontal axis.

Circuit Training – Polar Functions **Name________________________**

Directions: Beginning in the first cell marked #1, find the requested information. To advance in the circuit, hunt for your answer and mark that cell #2. Continue working in this manner until you complete the circuit. For cells marked non-calculator, find the exact answer yourself, THEN find the decimal equivalent to advance.

__1__ Ans: -0.302

Let $r(\theta) = 2\theta + \cos\theta$. Find the value of $\frac{dx}{d\theta}$ at $\theta = \pi/6$.

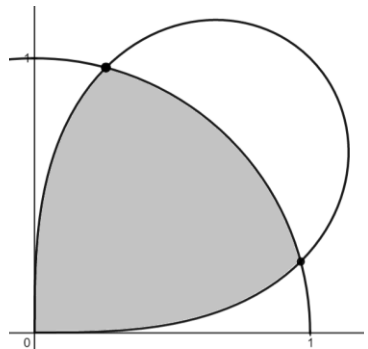


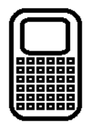
Exact answer: $\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} =$ ________________

To advance in the circuit, find the decimal approximation of the answer.

______ Ans: 62.013

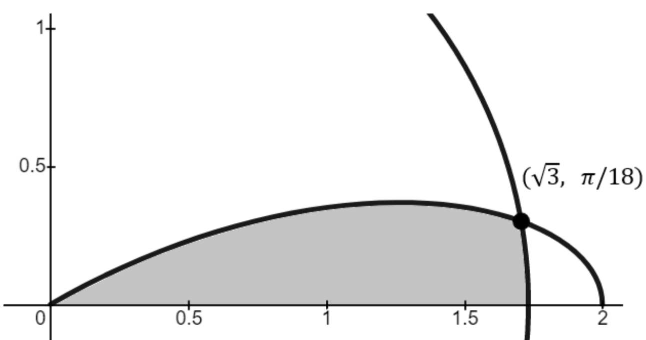
Find the area bounded by the circle $r(\theta) = 1$ and the curve $r(\theta) = \sqrt{2\sin(2\theta)}$ as shown.

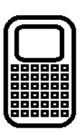




______ Ans: 0.205

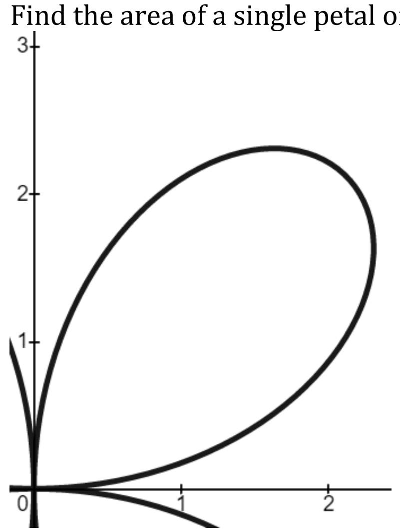
Find the area bounded as shown by $r(\theta) = 2\cos(3\theta)$ and $r = \sqrt{3}$.

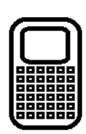




______ Ans: 7.029

Find the area of a single petal of $r(\theta) = 3\sin(2\theta)$.





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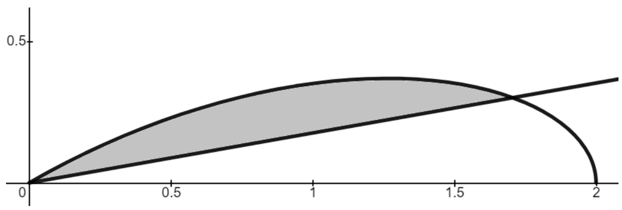
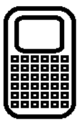
Circuit Training – Polar Functions

_______ Ans: 3.534

Find the area bounded as shown by $r(\theta) = 2\cos(3\theta)$ and $\theta = \pi/18$.





_______ Ans: 0.658

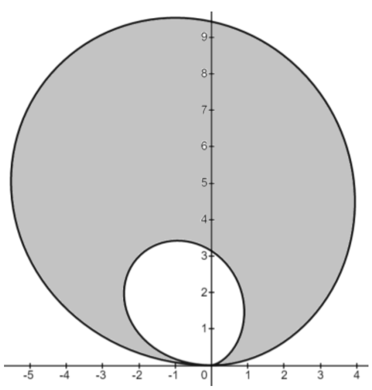
A particle is moving along the curve $r(\theta) = \theta + \sin\theta$ with $\frac{d\theta}{dt} = \frac{1}{3}$. Find $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}$.

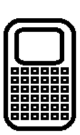
Exact answer: $\left.\frac{dx}{dt}\right|_{\theta=\pi/3}$ =________________

To advance in the circuit, find the decimal approximation of the answer.

_______ Ans: 0.467

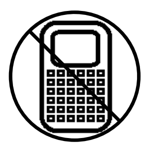
Find the area between the outer loop and inner loop of $r = 2\theta\sin\theta$ $[0, 2\pi]$.





_______ Ans: 0.342

Let $r(\theta) = 2\theta + \cos\theta$. Find the value of $\frac{dy}{dx}$ at $\theta = \pi/6$.



Exact answer: $\left.\frac{dy}{dx}\right|_{\theta=\pi/6}$ =________________

To advance in the circuit, find the decimal approximation of the answer.

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1: $\frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.342$

2: $\frac{\left(\frac{\sqrt{3}}{6}\pi + \frac{3}{2}\right)}{\frac{\sqrt{3}}{2} - \frac{\pi}{6}} \approx 7.029$

3: 3.534

4: 0.2047

5: 0.4665

6: 62.0125

7: 0.6575

8: $-\frac{\pi\sqrt{3}}{18} \approx -0.302$